Tutorial notes - 10 whizkiddeng@gmail.com

Indefinite integral.

substitution : try to transform the integral to some simpler form or familiar, basic form.

Q1. $\int \frac{dx}{x\sqrt{x^2-1}}$

(1) $\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{x\,dx}{x^2\sqrt{x^2-1}}$ (notice $d(x^2-1) = 2x\,dx$)

$= \frac{1}{2}\int \frac{d(x^2-1)}{x^2\sqrt{x^2-1}}$ ($d(\sqrt{t}) = \frac{1}{2}\frac{dt}{\sqrt{t}}$, in here $\underline{t = x^2-1}$

$= \int \frac{d\sqrt{x^2-1}}{x^2}$ (At last we choose $y = \sqrt{x^2-1}$) ($x^2 = y^2+1$)

$= \int \frac{dy}{y^2+1} = \arctan y + C = \arctan(\sqrt{x^2-1}) + C$ ①

(2) for factor $\sqrt{x^2-1}$, we often use $x = \sec t$ which comes from $\underline{\sec^2 t = \tan^2 t + 1}$

so for $\sqrt{x^2+1}$, the choice is $\underline{x = \tan t}$

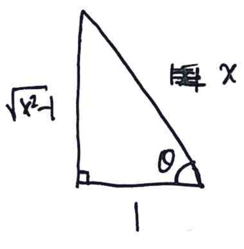
$\int \frac{dx}{x\sqrt{x^2-1}} \overset{x=\sec t}{=\!=} \int \frac{\sec t\cdot \tan t}{\sec t\,|\tan t|}dt$ ($dx = \sec t\cdot\tan t\,dt$) (the $|\tan t|$ comes from for $|x|>1$, means $t \in (0,\frac{\pi}{2}) \cup (\frac{\pi}{2},\pi)$, so $\tan t > 0$ in $(0,\frac{\pi}{2})$, and $\tan t < 0$ in $(\frac{\pi}{2},\pi)$, has different sign)

$= \pm\int dt = |t| + C$

$= |\arccos\frac{1}{x}| + C$ ② ($x = \sec t = \frac{1}{\cos t} \Rightarrow \cos t = \frac{1}{x} \Rightarrow t = \arccos\frac{1}{x}$)

compare the result ① and ②. we have different formula for a same integral. but actually they are the same answer, just different by some constants. We can see it from follows:

denote $\theta = \arctan(\sqrt{x^2-1}) \Rightarrow \tan\theta = \sqrt{x^2-1}$



so we can see $\cos\theta = \frac{1}{x} \Rightarrow \theta = \arccos\frac{1}{x}$

this is just the answer in ②. the absolute value just means we may have a constant pi.

similarly we can compute $\int \frac{dx}{x\sqrt{x^2+1}}$

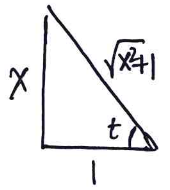
Q2 $\int \frac{dx}{x\sqrt{x^2+1}} \overset{x=\tan t}{=\!=\!=} \int \frac{\sec^2 t\, dt}{\tan t \cdot \sec t} = \int \frac{1}{\sin t}\, dt$

$$= \int \frac{\sin t}{\sin^2 t}\, dt = -\int \frac{d\cos t}{1-\cos^2 t} = -\int \left(\frac{1}{1-\cos t} + \frac{1}{1+\cos t}\right)\frac{1}{2}\, d\cos t$$

$$= -\frac{1}{2}\left(\ln|1+\cos t| - \ln|1-\cos t|\right) + C$$

$$= -\frac{1}{2}\ln\left|\frac{1+\cos t}{1-\cos t}\right| + C = -\frac{1}{2}\ln\left|\frac{1+\frac{1}{\sqrt{x^2+1}}}{1-\frac{1}{\sqrt{x^2+1}}}\right| + C$$

$$= -\ln(1+\sqrt{x^2+1}) - \ln|x| + C$$



$\tan t = x \Rightarrow \cos t = \frac{1}{\sqrt{x^2+1}}$

Q3. $\int \frac{x+3}{\sqrt{4x^2+4x+3}}\, dx.$

notice that $d(4x^2+4x+3) = (8x+4)\, dx$, then we try to construct such term.

$$= \frac{1}{8}\int \frac{8x+4}{\sqrt{4x^2+4x+3}}\, dx + \frac{5}{2}\int \frac{1}{\sqrt{4x^2+4x+3}}\, dx \quad (t = 4x^2+4x+3)$$

$$= \frac{1}{8}\int \frac{dt}{\sqrt{t}} + \frac{5}{2}\int \frac{1}{\sqrt{(2x+1)^2+2}}\, dx$$

$\frac{1}{8}\int \frac{dt}{\sqrt{t}} = \frac{1}{4}\sqrt{t}$ done

$\frac{5}{2}\int \frac{1}{\sqrt{(2x+1)^2+2}}\, dx \overset{t=2x+1}{=} \frac{5}{4}\int \frac{1}{\sqrt{t^2+2}}\, dt = \frac{5}{4}\ln(t+\sqrt{t^2+2}) + C$

such integral is $(\ln(x+\sqrt{x^2+a^2}))' = \frac{1}{\sqrt{x^2+a^2}}$

$\Rightarrow \int \frac{1}{\sqrt{x^2+a^2}}\, dx = \ln(x+\sqrt{x^2+a^2}) + C$

you can also prove it by using $x = a\tan t$

$$= \frac{1}{4}\sqrt{4x^2+4x+3} + \frac{5}{4}\ln(2x+1+\sqrt{4x^2+4x+3}) + C$$

Integrate by parts.

this is comes from the product-rule like:

$(uv)' = u'v + uv' \Leftrightarrow \int (uv)'\,dx = \int u'v\,dx + \int uv'\,dx$

$\Rightarrow uv = \int u'v\,dx + \int uv'\,dx \Rightarrow \underset{udv}{\int uv'\,dx} = uv - \underset{vdu}{\int u'v\,dx}$

the key is to find the factor u, v, and make the integral simpler.

Q4. $\int \ln(x+\sqrt{1+x^2})\,dx$

we take $u = \ln(x+\sqrt{1+x^2})$ $V = x$

So $= x\cdot\ln(x+\sqrt{1+x^2}) - \int x\,d\ln(x+\sqrt{1+x^2})$

$= x\cdot\ln(x+\sqrt{1+x^2}) - \underset{\int d\sqrt{1+x^2}}{\underset{\parallel}{\int \frac{x}{\sqrt{1+x^2}}\,dx}} = x\cdot\ln(x+\sqrt{1+x^2}) - \sqrt{1+x^2} + C$

Q5. $\int \frac{x e^{\arctan x}}{(1+x^2)^{3/2}}\,dx$

First use the substitution $t = \arctan x \Leftrightarrow x = \tan t$

$= \int \frac{\tan t\, e^t}{(1+\tan^2 t)^{3/2}} \sec^2 t\,dt = \int e^t \sin t\,dt$ much simpler

$I = \int \sin t\,de^t$ (integrate by parts)

$= e^t\cdot\sin t - \int e^t\,d\sin t = e^t\sin t - \int e^t\cos t\,dt$, $\underline{I = e^t\sin t - J}$ (1)

denote $J = \int e^t\cos t\,dt = \int \cos t\,de^t = e^t\cos t - \int e^t\,d\cos t$

$= e^t\cos t + \int e^t\sin t\,dt$

$= e^t\cos t + I$ (2)

Combine (1) (2) we get $I = \frac{1}{2}e^t(\sin t - \cos t) + C$. back to x

Q6. $\int \sqrt{1+x^2}\,dx$

(1) integrate by parts

$$I = \int \sqrt{1+x^2}\, d(x) = x\sqrt{1+x^2} - \int x\, d\sqrt{1+x^2}$$

$$= x\sqrt{1+x^2} - \int \frac{x^2}{\sqrt{1+x^2}}\,dx = x\sqrt{1+x^2} - \int \frac{(x^2+1)-1}{\sqrt{x^2+1}}\,dx$$

$$= x\sqrt{1+x^2} - \underbrace{\int\sqrt{1+x^2}\,dx}_{I} + \underbrace{\int\frac{1}{\sqrt{1+x^2}}\,dx}_{= \ln(x+\sqrt{1+x^2})}$$

$$\Rightarrow I = \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\ln(x+\sqrt{1+x^2}) + C$$

(2) use $x = \tan t$

$$\int\sqrt{1+x^2}\,dx = \int \sec t \cdot \sec^2 t\,dt = \int \frac{1}{\cos^3 t}\,dt = \int \frac{\cos t}{\cos^4 t}\,dt \quad (y = \sin t)$$

$$= \int \frac{dy}{(1-y^2)^2} = \int \frac{dy}{(1-y)^2(1+y)^2}$$ this is what we called rational function.

for $\frac{1}{(1-y)^2(1+y)^2} = \frac{A}{1-y} + \frac{B}{(1+y)} + \frac{C}{(1-y)^2} + \frac{D}{(1+y)^2}$ we can have such decomposition, A, B, C, D are constants we need determiont.

Now I just tell you $A = B = C = D = \frac{1}{4}$.

$$\Rightarrow \int\sqrt{1+x^2}\,dx = \frac{1}{4}\int\left(\frac{1}{1-y} + \frac{1}{1+y} + \frac{1}{(1-y)^2} + \frac{1}{(1+y)^2}\right)dy$$

$$= \frac{1}{4}\ln\left|\frac{1+y}{1-y}\right| + \frac{1}{4}\frac{1}{1-y} - \frac{1}{4}\frac{1}{1+y} + C$$

Math 1010C   Tutorial - 10   13/11/2014   Jiang Qingyuan

PLAN: Indefinite integral;

Basic techniques: (i) substitution;
(ii) Integration by parts; ← focus on these two
(iii) by trigonometric identities (trig substitution; using trig formulas; reduction formulas)

---

(0) Basic ones: TRY to write down the answer without refering to your text book!

(1) $\int x^\alpha dx = \frac{1}{\alpha+1} x^{\alpha+1} + C, \quad (\alpha \neq -1)$

(2) $\int \frac{dx}{x} = \ln|x| + C$;

(3) $\int e^x dx = e^x + C$;

(4) $\int \cos x \, dx = \sin x + C$;

(5) $\int \sin x \, dx = \cos x + C$;

(6) $\int \frac{dx}{\cos^2 x} = \tan x + C$;
($\int \frac{dx}{\sin^2 x} = -\cot x + C$)

(7) $\int \frac{dx}{1+x^2} = \arctan x + C$;

(8) $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$;

(9) $\int \sinh x \, dx = \cosh x + C$;

(10) $\int \cosh x \, dx = \sinh x + C$;

(11) $\int \frac{dx}{\cosh^2 x} = \tanh x + C$;

. . . . (text book, page 59)

Basic properties:

$\int (f \pm g) = \int f \pm \int g$;

$\int kf = k \int f$;

Integration by parts:

$\int u \, dv = uv - \int v \, du$;

Basic trigonometric identities:

- $\sin^2 x + \cos^2 x = 1$;
- $\frac{1}{\cos^2 x} = 1 + \tan^2 x$;
- $\sin 2x = 2 \sin x \cos x$,
- $\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \cos^2 x - \sin^2 x$;
- $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$

(For more, see text book, page 82)

(i) Basic teichniques, & Basic subtitution

① $\int \frac{x^4}{1+x^2}\,dx$

Sol'n: $\int \frac{x^4}{1+x^2}\,dx = \int \frac{x^4-1+1}{1+x^2}\,dx = \int (x^2-1+\frac{1}{1+x^2})\,dx = \frac{x^3}{3} - x + \arctan x + C$;

② $\int \tan^2 x\,dx$

Sol'n: $\int \tan^2 x\,dx = \int \frac{\sin^2 x}{\cos^2 x}\,dx = \int \left(\frac{1}{\cos^2 x} - 1\right)dx = \tan x - x + C$;

③ $\int \frac{1}{\sin^2 2x}\,dx$

Sol'n: $\int \frac{1}{\sin^2 2x}\,dx = \int \frac{1}{4\sin^2 x\cos^2 x}\,dx = \int \frac{\sin^2 x + \cos^2 x}{4\sin^2 x\cos^2 x}\,dx = \frac{1}{4}\left(\int \frac{1}{\cos^2 x}\,dx + \int \frac{1}{\sin^2 x}\,dx\right)$

$= \frac{1}{4}(\tan x - \cot x) + C$;

④ $\int \frac{dx}{ax+b}$ $(a \neq 0)$.

Sol'n: Let $u = ax+b$. $du = a\,dx$ i.e $dx = \frac{1}{a}du$. then

$\int \frac{dx}{ax+b} = \int \frac{1}{a}\frac{du}{u} = \frac{1}{a}\ln|u| + C = \frac{1}{a}\ln|ax+b| + C$;

⑤ $\int (ax+b)^n\,dx$ $(a \neq 0, n \neq -1)$.

Sol'n: Let $u = ax+b$. $du = a\,dx$, i.e. $dx = \frac{1}{a}du$;

$\int (ax+b)^n\,dx = \frac{1}{a}\int u^n\,du = \frac{u^{n+1}}{a(n+1)} + C = \frac{(ax+b)^{n+1}}{a(n+1)} + C$;

⑥ $\int \frac{dx}{a^2+x^2}$ $(a \neq 0)$

Sol'n: since $\int \frac{dx}{a^2+x^2} = \frac{1}{a^2}\int \frac{dx}{1+(\frac{x}{a})^2}$, Let $u = \frac{x}{a}$, $dx = a\,du$;

Then $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\int \frac{du}{1+u^2} = \frac{1}{a}\arctan\frac{x}{a} + C$;

⑦ $\int \frac{dx}{x^4-1}$.

Sol'n: $\int \frac{dx}{x^4-1} = \int \frac{dx}{(x^2-1)(x^2+1)} = \frac{1}{2}\left(\int \frac{dx}{x^2-1} - \int \frac{dx}{x^2+1}\right)$

$= \frac{1}{2}\left[\frac{1}{2}\left(\int \frac{dx}{x-1} - \int \frac{dx}{x+1}\right) - \int \frac{dx}{x^2+1}\right]$

$= \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\arctan x + C$;

⑧ $\int \frac{dx}{a^2-x^2}$ $(a\neq 0)$

Sol'n: $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\int \left(\frac{1}{a+x} - \frac{1}{a-x}\right)dx = \frac{1}{2a}\left[\int \frac{d(x+a)}{x+a} - \int \frac{d(x-a)}{x-a}\right]$

$= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$;

⑨. $\int \frac{dx}{\sqrt{x(1-x)}}$;

Sol'n: Method 1 | $\int \frac{dx}{\sqrt{x(1-x)}} = \int \frac{dx}{\sqrt{\frac{1}{4}-(x-\frac{1}{2})^2}} = \arcsin(2x-1) + C$;

method 2 | let $u=\sqrt{x}$, $du = \frac{dx}{2\sqrt{x}}$ or $dx = 2\sqrt{x}\,du$

$\int \frac{dx}{\sqrt{x(1-x)}} = 2\int \frac{du}{\sqrt{1-u^2}} = 2\arcsin(\sqrt{x}) + C$;

Rmk: Both results are correct! Actually they differ by a const!

⑩. $\int \tan x$.

Sol'n: since $\sin x\,dx = -d\cos x$; $\int \tan x = -\int \frac{d\cos x}{\cos x} = -\ln|\cos x| + C$;

(ii) Integration by parts : $\int u\,dv = uv - \int v\,du$;

Methods : express the form in terms of uv', where $v' = e^{\pm x}$, $\sin x$, $\cos x$, x^n, $\sinh x$, $\cosh x$, etc.

Then use integration by parts to get simpler integrals, (or equations of the original integral, see later)

① $\int \arctan x \, dx$.

Sol'n : $v = x$, $u = \arctan x$; $v' = 1$. $uv' = \arctan x$;

$$\int \arctan x \, dx = x \arctan x - \int x \, \underbrace{d(\arctan x)}_{\frac{dx}{1+x^2}}$$

$$= x\arctan x - \int \frac{x\,dx}{1+x^2} = x\arctan x - \frac{1}{2}\ln(1+x^2) + C.$$

② $\int x^2 \sin x \, dx$. (very similar to text book e.g. $\int x^2 \cos x$)

Sol'n : $v = \cos x$, $u = x^2$, $(uv' = -x^2 \sin x$;)

$$\int x^2 \sin x \, dx = -\int x^2 d(\cos x) = -x^2\cos x + \int \cos x \, d(x^2)$$

$$= -x^2\cos x + 2\int \overbrace{x\cos x\,dx}^{x\, d(\sin x)}$$

use again!

$$= -x^2\cos x + 2x\sin x - 2\int \sin x \, dx$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + C;$$

③ $\int x^3 \ln^2 x \, dx$.

Sol'n : $$\int \ln^2 x \, d\frac{x^4}{4} = \frac{1}{4}x^4\ln^2 x - \frac{1}{4}\int x^4 \, d\ln^2 x = \frac{1}{4}x^4\ln^2 x - \frac{1}{2}\int x^3 \ln x \, dx$$

now the problem is turned into examples in text book $\int x^3 \ln x \, dx$.

$$\int x^3 \ln^2 x\, dx = \frac{1}{4}x^4 \ln^2 x - \frac{1}{2}\underbrace{\left(\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4\right)} + C$$

$$= \frac{1}{4}x^4\left(\ln^2 x - \frac{1}{2}\ln x + \frac{1}{8}\right) + C.$$

[recall: $\int x^3 \ln x\, dx = \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3 dx = \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C.$
$\left(u = \ln x, \ v = x^4, \ dv = x^3 dx; \right)$]

Rmk: All these types of integrals can be computed using integration by parts:

$$\int x^k \ln^m x\, dx. \quad \int x^k \sin bx\, dx. \quad \int x^k \cos bx\, dx, \quad \int x^k e^{ax} dx,$$

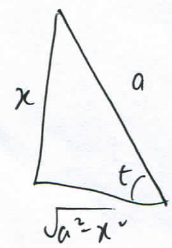
(iii) First, trignometric substitution:

① $\int \sqrt{a^2-x^2}\, dx$

Sol'n: $x = a\sin t$, $|t| < \frac{\pi}{2}$, then

$$\int \sqrt{a^2-x^2}\, dx = \int a^2\cos^2 t\, dt = \frac{a^2}{2}t + \frac{a^2}{4}\sin 2t + C$$

$$= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{1}{2}x\sqrt{a^2-x^2} + C;$$



② $I = \int \frac{1}{\sqrt{x^2-a^2}}\, dx \quad (a>0, \ |x|>a).$

Sol'n: (i) $x = a\sec t = \frac{a}{\cos t} \quad (0 < t < \frac{\pi}{2}).$

then $I = \int \frac{a\sec t\tan t}{a\tan t}\, dt = \underbrace{\int \frac{dt}{\cos t} = \ln|\sec t + \tan t| + C}$

$= \ln|x + \sqrt{x^2-a^2}| + C.$

[$\cos t\, dt = d\sin t$
$\int \frac{dt}{\cos t} = \int \frac{\cos t\, dt}{\cos^2 t} = \int \frac{d\sin t}{1-\sin^2 t}$
$= \frac{1}{2}\ln\left|\frac{1+\sin t}{1-\sin t}\right| + C = \ln|\sec t + \tan t| + C.$]

补10-3

③ $\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx.$

Sol'n: $x = t^6$;

$$I = \int \frac{6t^5\,dt}{t^3+t^2} = 6\int \frac{t^3}{1+t}\,dt = 6\int \left(t^2 - t + 1 - \frac{1}{1+t}\right)dt$$

$$= \cancel{3t}\ 2t^3 - 3t^2 + 6t - 6\ln|1+t| + C$$

$$= 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln|1+\sqrt[6]{x}| + C.$$

[Further Techniques!]

(iv) Reduction: (Reduce to sth "easier", by induction get results or get relations of different integrals)

① $I = \int e^{ax}\cos bx\,dx$, $\quad J = \int e^{ax}\sin bx\,dx$;

Sol'n:

$$\begin{cases} I = \frac{1}{b}\int e^{ax}\,d(\sin bx) = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}J; \\ J = -\frac{1}{b}\int e^{ax}\,d(\cos bx) = -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}I; \end{cases}$$

solve eq'ns:

$$I = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2} + C;$$

$$J = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} + C;$$

② $K_n = \int \cos^n x\,dx$; ([Fikhtengol'ts])

Sol'n: $K_n = \int \cos^{n-1}x\,d(\sin x) = \sin x\cdot\cos^{n-1}x - \int \sin x\,d(\cos^{n-1}x)$

$$= \sin x\cdot\cos^{n-1}x + (n-1)\int \underbrace{\sin^2 x}_{=(1-\cos^2 x)}\cdot\cos^{n-2}x\,dx$$

$$= \sin x\cdot\cos^{n-1}x + (n-1)K_{n-2} - (n-1)K_n.$$

Hence $K_n = \frac{1}{n} \sin x \cdot \cos^{n-1} x + \frac{n-1}{n} K_{n-2}$;

Hence $K_0 = \int dx = x + C$;

$K_1 = \int \cos x \, dx = \sin x + C$;

$K_2 = \frac{1}{2} \sin x \cdot \cos x + \frac{1}{2} K_0 + C = \frac{1}{4} \sin 2x + \frac{1}{2} x + C$;

$K_3 = \frac{1}{3} \sin x \cdot \cos^2 x + \frac{2}{3} K_1 + C = \frac{1}{3} \sin x \cdot \cos^2 x + \frac{2}{3} \sin x + C$;

......

Rmk: textbook ; $I_n = \int \sin^n x \, dx$.

Get $I_n = -\int \sin^{n-1} x \, d\cos x = -\sin^{n-1} \cos x + (n-1) \int \cos^2 x \sin^{n-2} x \, dx$

$= -\sin^{n-1} x \cdot \cos x + (n-1) I_{n-2} - (n-1) I_n$;

$\Rightarrow I_n = -\frac{\sin^{n-1} x \cdot \cos x}{n} + \frac{n-1}{n} I_{n-2}$;

③ $I_{k,m} = \int x^k \ln^m x \, dx$; ($\overset{\text{real number}}{k \neq -1}$, $m \in \mathbb{Z}_{\geq 0}$) ([textbook])

Sol'n: $\underbrace{\int x^k \ln^m x \, dx}_{I_{k,m}} = \int \ln^m x \cdot d\frac{x^{k+1}}{k+1}$

$= \frac{1}{k+1} x^{k+1} \ln^m x - \frac{m}{k+1} \underbrace{\int x^k \ln^{m-1} x \, dx}_{I_{k,m-1}}$

& $I_{k,0} = \int x^k dx = \frac{x^{k+1}}{k+1} + C$. $\Rightarrow$ Known $I_{k,m}$, $\forall m > 0$ integer.

#

④ $J_n = \int \frac{dx}{(x^2+a^2)^n}$, $(n=1,2,3,\dots)$ ([Fikhtengol'ts])

Sol'n: $u = \frac{1}{(x^2+a^2)^n}$, $dv = dx$; $\underline{du = -\frac{2nx}{(x^2+a^2)^{n+1}}\,dx}$

$$J_n = \frac{x}{(x^2+a^2)^n} + 2n\underbrace{\int \frac{x^2}{(x^2+a^2)^{n+1}}\,dx}_{= \, J_n - a^2 J_{n+1}}$$

$(x^2+a^2)-a^2$

Get: $J_{n+1} = \frac{1}{2na^2}\frac{x}{(x^2+a^2)^m} + \frac{2n-1}{2n}\cdot\frac{1}{a^2}J_n.$

Then get

$$J_1 = \frac{1}{a}\arctan\frac{x}{a};$$

$$J_2 = \frac{1}{2a^2}\frac{x}{x^2+a^2} + \frac{1}{2a^3}\arctan\frac{x}{a};$$

$$J_3 = \frac{1}{4a^2}\frac{x}{(x^2+a^2)^2} + \frac{3}{4a^2}J_2 = \cdots;$$

etc.

□

Tutorial 10

Topics : Indefinite Integral

Q1) Evaluate the indefinite integral

a) $\int x e^{-x^2} dx$ b) $\int \frac{1}{x^2 - 2x - 3} dx$

Q2) Evaluate the integral by u-substitution

a) $\int x e^{-x} dx$ b) $\int \frac{x^n}{\sqrt{x-1}} dx$; n: positive integer.

Q3) Evaluate the integral by trigonometric substitution.

a) $\int \sin^5 x \, dx$ b) $\int \sin^{-1} x \, dx$

Q4) Evaluate $I_n = \int x^n e^{ax} dx$ by reduction method for $n \in \mathbb{N}$.

& evaluate I_2 explicitly.

Recall:

① <u>Substitution method</u> : Suppose f, u are smooth functions.

Then $\int f(x)\, dx = \int f(y(t))\, y'(t)\, dt$

② <u>Integration by parts</u> : Suppose f, g are smooth functions

Then $\int f(x)\, d\, g(x) = f(x) g(x) - \int g(x)\, d f(x)$

[in short , $\int f dg = f g - \int g df$]

③ <u>Reduction method</u> Suppose f_n be smooth function $\forall n \in \mathbb{N}$.

If $I_n := \int f_n(x)\, dx = \cdots = I_{n-1} + F_n(x)$ for some known function F_n.

Then inductively I_n can be evaluated (theoretically) if I_0 is known.

$\underline{\text{Sol}^n}$

Q1a)

$$\int x e^{-x^2} dx = \int e^{-x^2} d\frac{x^2}{2} = \frac{-1}{2}\int e^{(-x^2)} d(-x^2)$$

$$= \frac{-1}{2} e^{-x^2} + C \qquad \exists c \in \mathbb{R}$$

Q1b)

$$\int \frac{1}{x^2 - 2x - 3} dx = \int \frac{1}{(x+1)(x-3)} dx$$

$$= \int \frac{1}{4}\left(\frac{-1}{x+1} + \frac{1}{x-3}\right) dx = \frac{-1}{4}\int \frac{dx}{x+1} + \frac{1}{4}\int \frac{dx}{x-3}$$

$$= \frac{-1}{4} \ln|x+1| + \frac{1}{4}\ln|x-3| + C \qquad \exists C \in \mathbb{R}$$

Q2a)

$$\int xe^{-x}\,dx$$

Sub $y = e^{-x} \Rightarrow x = -\ln y$

& $dx = \frac{-1}{y}dy$

$$= \int (-\ln y)(y)\left(\frac{-1}{y}\,dy\right)$$

$$= \int \ln y\,dy = y\ln y - \int y\,d\ln y$$

$$= y\ln y - \int dy = y\ln y - y + c \qquad \exists\, c \in \mathbb{R}$$

$$= -xe^{-x} - e^{-x} + c$$

Q2b)

$$\int \frac{x^n}{\sqrt{x-1}}\,dx = \int \frac{(y^2+1)^n}{y}(2y\,dy)$$

$$\left[\begin{array}{l} \text{Sub } y = \sqrt{x-1} \\ \Rightarrow x = y^2+1 \\ \Rightarrow dx = 2y\,dy \end{array}\right]$$

$$= 2\int (y^2+1)^n\,dy = 2\int \sum_{r=0}^{n} {}^{n}C_r\, y^{2r}\,dy$$

$$= \sum_{r=0}^{n} \frac{2n!\,y^{2r+1}}{r!(n-r)!(2r+1)} + c = \sum_{r=0}^{n} \frac{2n!}{r!(n-r)!(2r+1)}(x-1)^{r+\frac{1}{2}} + C$$

//

Q3a) $\int \sin^5 x \, dx$

Sub $y = \cos x$, $dy = -\sin x \, dx$

& $\sin^2 x = 1 - \cos^2 x = 1 - y^2$

$= \int (\sin^2 x)^2 (\sin x \, dx)$

$= \int (1-y^2)^2 (-dy)$

$= \int -y^4 + 2y^2 - 1 \, dy = \frac{-1}{5} y^5 + \frac{2}{3} y^3 - y + C$, $\exists C \in \mathbb{R}$

$= \frac{-1}{5} \cos^5 x + \frac{2}{3} \cos^3 x - \cos x + C$ //

Q3b) $\int \sin^{-1}x \, dx$

Sub $y = \sin^{-1}x \Rightarrow x = \sin y$

$dx = \cos y \, dy$

$= \int y \cos y \, dy$

$= \int y \, d\sin y$

$= y \sin y - \int \sin y \, dy$

$= y \sin y + \cos y + c \qquad \exists c \in \mathbb{R}.$

$= x \sin^{-1}x + \cos(\sin^{-1}x) + c$ //

Q4) Let $n \in \mathbb{N}$, $I_n = \int x^n e^{ax}\, dx$; $a \neq 0$

$$I_n = \int x^n \frac{de^{ax}}{a} = \frac{x^n e^{ax}}{a} - \frac{1}{a}\int e^{ax}\, dx^n$$

$$= \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax}\, dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$$

Hence we have $I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$

In particular for $n = 2$.

$$I_2 = \frac{x^2 e^{ax}}{a} - \frac{2}{a} I_1 = \frac{x^2 e^{ax}}{a} - \frac{2}{a}\left(\frac{x e^{ax}}{a} - \frac{I_0}{a}\right)$$

$$= \frac{x^2 e^{ax}}{a} - \frac{2 e^{ax} x}{a^2} + \frac{2}{a^2}\int e^{ax}\, dx$$

$$= e^{ax}\left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right) + C \quad , \exists C \in \mathbb{R} \quad //$$

Techniques to compute Indefinite integrals

① usual formula for functions like sin, cos, etc.

Ex: (1) $\int \frac{dx}{1-\cos x}$

$= \int \frac{dx}{2\sin^2\frac{x}{2}}$ $\quad (1-\cos x = 2\sin^2\frac{x}{2})$

$= \int \frac{d(\frac{x}{2})}{\sin^2\frac{x}{2}}$

$= \int \csc^2 t\, dt \quad (t=\frac{x}{2})$

$= -\cot\frac{x}{2} + C \quad ((\cot x)' = -\csc^2 x)$

(2) $\int \sin 3x \sin 5x\, dx$

$= \frac{1}{2}\int [\cos(5x-3x) - \cos(5x+3x)]\,dx$ $\quad (\cos(5x-3x) = \cos 5x\cos 3x + \sin 5x \sin 3x,$
$\cos(5x+3x) = \cos 5x \cos 3x - \sin 5x \sin 3x)$

$= \frac{1}{2}\int \cos 2x\, dx - \int \cos 8x\, dx$

$= \frac{1}{4}\sin 2x - \frac{1}{16}\sin 8x + C$

② Integration by parts.

$\int f(x)\,dg(x) = f(x)g(x) - \int g(x)\,df(x)$

Ex: (1) $\int \ln x\, dx$

$= x\ln x - \int x \cdot \frac{1}{x}\, dx$

$= x\ln x - x + C$

(2) $\int (\frac{\ln x}{x})^2 dx$

$= \int \frac{(\ln x)^2}{x^2} dx = \int \frac{(\ln x)^2}{x} d(\ln x)$

$= \int \frac{t^2}{e^t} dt \quad (t = \ln x)$

$= \int t^2 e^{-t} dt = -\int t^2 d(e^{-t})$

$= -(t^2 e^{-t} - 2\int te^{-t} dt)$

$= 2\int te^{-t} dt - t^2 e^{-t}$

$= -2(te^{-t} - \int e^{-t} dt) - t^2 e^{-t}$

$= -2e^{-t} - 2te^{-t} - t^2 e^{-t} + C$

③ Substitution.

EX: 1) $\int \sin(\ln x) dx$

~~$\int \sin t dt$~~

$= \int e^t \sin t \, dt \quad (t = \ln x \quad dt = \frac{1}{x} dx)$

$= \int \sin t \, de^t$

$= e^t \sin t - \int \cos t \cdot e^t dt$

$= e^t \cdot \sin t - (\int \cos t \, de^t)$

$= e^t \cdot \sin t - (\cos t \cdot e^t + \int e^t \sin t \, dt)$

$\int \sin t \cdot e^t dt = \frac{1}{2} e^t (\sin t - \cos t) + C$

④ Recurrence

ex: (1) $I_n = \int x^n e^{ax} dx$

$$\int x^n e^{ax} dx$$
$$= \frac{1}{a}\int x^n d e^{ax}$$
$$= \frac{1}{a}(x^n e^{ax} - n\int x^{n-1} \cdot e^{ax} dx)$$
$$= \frac{1}{a} x^n e^{ax} - \frac{n}{a} I_{n-1}$$

Then $I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$

若 $n=1$, $I_1 = \int x e^{ax} dx = \frac{1}{a}\int x \, de^{ax} = \frac{1}{a}(xe^{ax} - \int e^{ax} dx)$
$$= \frac{1}{a}(xe^{ax} - \frac{1}{a} e^{ax} + C_1)$$
$$= \frac{e^{ax}}{a}(x-1) + C_2$$

Exercise:

(1) $\int \cos^3 x \, dx$ (2) $\int \frac{dx}{\cos x \sin^2 x}$ (3) $\int \cos x \cos 2x \cos 3x \, dx$

(4) $\int \sin^{-1} x \, dx$ (5) $\int x \sin^2 x \, dx$ (6) $\int \ln(x + \sqrt{1+x^2}) dx$

(7) $I_n = \int \sin^n x \, dx$ $n \geq 2$